

# Towards A Background Independent Quantum Gravity

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**Abstract.** We recapitulate the scheme of emergent gravity to highlight how a background independent quantum gravity can be defined by quantizing spacetime itself.

## 1. Introduction

According to general relativity, gravity is the dynamics of spacetime geometry where spacetime is a (pseudo-)Riemannian manifold and the gravitational field is represented by a Riemannian metric [1]. The gravitational field equations are given by the Einstein equations defined by

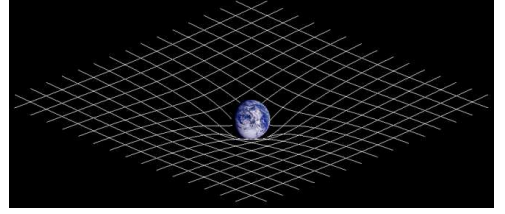
$$R_{MN} - \frac{1}{2}g_{MN}R = \frac{8\pi G}{c^4}T_{MN}. \quad (1)$$

A beauty of the equation (1) may be phrased in a poetic diction (John A. Wheeler) that matter tells spacetime how to curve, and spacetime tells matter how to move. However the notorious difficulty to elevate the intimate cooperation between spacetime geometry and matters to a quantum world has insinuated doubt into us for the harmonious conspiracy. Therefore we will take a closer look at (1) aiming to reveal an inmost conflict in (1) behind the superficial harmony.

The first observation [2] is that the gravitation described by (1) presupposes a physically inviable vacuum. If one consider a flat spacetime whose metric is given by  $g_{MN} = \eta_{MN}$ , the left-hand side of (1) identically vanishes and so the right-hand side must cultivate a completely empty space with  $T_{MN} = 0$ . But the concept of empty space in Einstein gravity is in acute contrast to the concept of vacuum in quantum field theory (QFT) where the vacuum is not empty but full of quantum fluctuations. As a result, QFT claims that such an empty space of nothing is inviable in Nature and, instead, the vacuum is extremely heavy whose weight is maximally of Planck mass, i.e.,  $\rho_{\text{vac}} \sim M_P^4$ . Thus, if there is no way to completely suppress quantum fluctuations, Einstein gravity probably either presupposes a physically inviable space or incorrectly identifies the genetic origin of flat spacetime.

The second observation [3] is that flat spacetime in general relativity behaves like an elastic body with tension although the flat spacetime itself is the geometry of special relativity. We have seen that slides for colloquium introducing general relativity to the public or undergraduate students often contain the image like Figure 1 below. The Figure 1 illustrates how a massive body changes the geometry of spacetime around the mass point. In general relativity, this (curved) geometry is described by the gravitational field  $g_{MN}(x) = \eta_{MN} + h_{MN}(x)$  and interpreted as gravity. When the massive body moves to another place, the original point

where the body was placed will recover a (nearly) flat geometry like a rubber band. That is, the (flat) spacetime behaves like a metrical elasticity which opposes the curving of space. But this picture rather exhibits a puzzling nature of flat spacetime because the flat spacetime should be a completely empty space without any kind of energy as we remarked above. How is it possible for an empty space of nothing to behave like an elastic body with tension? Moreover we know that the gravitational force is extremely weak which implies that the space strongly withstands the curving and so the tension of spacetime would be extremely big, maybe, of the Planck energy.



**Figure 1.** Two-dimensional analogy of spacetime distortion. (Image from Wikipedia: Spacetime)

The third observation [4] is that the gravitational field  $g_{MN}(x) = \eta_{MN} + h_{MN}(x)$  has a vacuum expectation value (vev), i.e.,  $\langle g_{MN}(x) \rangle_{\text{vac}} = \eta_{MN}$  like the Higgs field  $\phi(x) = v + h(x)$ . These two fields also describe particles (either spin-2 graviton or spin-0 Higgs) as usual quantum fields in Standard Model. However these two particles are very exotic because all other fields, denoted as  $\Psi$ , in Standard Model have a zero vev;  $\langle \Psi \rangle_{\text{vac}} = 0$ . We are reasonably understanding why the Higgs field has the nonzero vev which triggers the electroweak symmetry breaking. In effect, Standard Model is defined in a nontrivial vacuum with the Higgs condensate  $v = \langle \phi \rangle_{\text{vac}}$  whose dynamical scale is around  $v \sim 246$  GeV. Therefore any particle interacting with the Higgs field feels a resistance in vacuum and acquires a mass. What about the flat spacetime  $\eta_{MN} = \langle g_{MN}(x) \rangle_{\text{vac}}$ ? Is it also originated from some kind of vacuum condensate? If so, what is the dynamical scale of the condensate? Note that the gravitation is characterized by its own intrinsic scale given by the Newton constant  $G = L_P^2$  where  $L_P = M_P^{-1} \sim 10^{-33}$  cm is the Planck length and classical gravity corresponds to  $L_P \rightarrow 0$  limit.

The fourth observation [5, 6] is that gravity may be not a fundamental force but an emergent force. A well-known fact is that graviton is to a messenger particle of the gravitational force as photon is to a messenger particle of the electromagnetic force. Hence the graviton will not be a fundamental particle either, if gravity is not a fundamental force. In general relativity the gravitational force is represented by a Riemannian metric of curved spacetime manifold  $M$

$$\left(\frac{\partial}{\partial s}\right)^2 = g^{MN}(x) \frac{\partial}{\partial x^M} \otimes \frac{\partial}{\partial x^N}. \quad (2)$$

It is well-known that the metric (2) in the tetrad formalism can be defined by the tensor product of two vector fields  $E_A = E_A^M(x) \frac{\partial}{\partial x^M} \in \Gamma(TM)$  as follows

$$\left(\frac{\partial}{\partial s}\right)^2 = \eta^{AB} E_A \otimes E_B. \quad (3)$$

Mathematically, a vector field  $X$  on a smooth manifold  $M$  is a derivation of the algebra  $C^\infty(M)$ . Here the vector fields  $E_A \in \Gamma(TM)$  are the smooth sections of tangent bundle  $TM \rightarrow M$  which are dual to the vector space  $E^A = E_A^M(x) dx^M \in \Gamma(T^*M)$ , i.e.,  $\langle E^A, E_B \rangle = \delta_B^A$ . The expression (3) glimpses the avatar of gravity that a spin-two graviton might arise as a composite of two spin-one *vector fields*. In other words, the tensor product (3) can be abstracted by the relation  $(1 \otimes 1)_S = 2 \oplus 0$ . Note that any field  $\Psi$  for fundamental particles in Standard Model cannot be written as the tensor product of other two fields, so to say,  $\Psi = \Psi_1 \otimes \Psi_2$ . Only composite particles (or bound states) such as mesons can be represented in such a way. Therefore graviton represented by the tensor product (3) is certainly different from fundamental particles in Standard Model.

The final observation [7, 8] is that there is an acute mismatch of symmetry between gravity and matters because gravity is the only interaction sensitive to a shift of the Lagrangian by an additive constant. To be precise, if one shift a matter Lagrangian  $\mathcal{L}_M$  by a constant  $\Lambda$ , that is,

$$\mathcal{L}_M \rightarrow \mathcal{L}'_M = \mathcal{L}_M - 2\Lambda, \quad (4)$$

it results in the shift of the energy-momentum tensor by  $T_{MN} \rightarrow T_{MN} - \Lambda g_{MN}$  in the Einstein equation (1) although the equations of motion for matters are invariant under the shift (4). Definitely the  $\Lambda$ -term in (4) will appear as the cosmological constant in Einstein gravity and it affects the spacetime structure. For instance, a flat spacetime is no longer a solution of (1). Even worse is that this clash of symmetry brings about the stability problem of spacetime. As we remarked in the first observation, the vacuum in QFT is a stormy sea of quantum fluctuations which accommodates the vacuum energy of the order of  $M_P^4$ . Fortunately the vacuum energy due to the quantum fluctuations, regardless of how large it is, does not make any trouble to QFT thanks to the symmetry (4). However the general covariance requires that gravity couples universally to all kinds of energy. Therefore the vacuum energy  $\rho_{\text{vac}} \sim M_P^4$  will induce a highly curved spacetime whose curvature scale  $R$  would be  $\sim M_P^2$  according to (1). If so, the QFT framework in the background of quantum fluctuations must be broken down due to a large back-reaction of background spacetime. But we know that it is not the case. QFT is well-defined as ever in the presence of the vacuum energy because the background spacetime still remains flat, as we empirically know. What is wrong with this argument ?

After consolidating all the suspicions inferred above, we throw a doubt on the genesis that flat spacetime is free gratis, i.e., costs no energy. All the above reasonings imply that the negligence about the dynamical origin of flat spacetime defining a local inertial frame in general relativity might be a core root of the incompatibility inherent in (1). It should be remarked that the genesis about spacetime cannot be addressed within the context of general relativity because flat spacetime is a geometry of special relativity rather than general relativity and so it is assumed to be *a priori* given without reference to its dynamical origin. All in all, it is tempted to infer that flat spacetime may be not free gratis but a result of Planck energy condensation in vacuum [9, 10]. Surprisingly, if that inference is true, it appears as the Hóly Gráil to cure several notorious problems in theoretical physics; for example, to resolve the cosmological constant problem, to understand the nature of dark energy and to explain why gravity is so weak compared to other forces. After all, the target is to formulate a background independent theory<sup>1</sup> to correctly explain the dynamical origin of flat spacetime. Note that Einstein gravity is not completely background independent since it assumes the prior existence of a spacetime manifold. But it turns out [11, 12, 5] that the emergent gravity from noncommutative (NC) geometry precisely realizes the desired property, as will be surveyed in the next sections.

## 2. Einstein gravity from electromagnetism on symplectic space

Now we will show that the vierbeins in (3) and so the Riemannian metrics arise from electromagnetic fields living in a space  $(M, B)$  supporting a symplectic structure  $B$  [13, 14, 15, 16].<sup>2</sup> See [17, 18, 19, 5, 20] for recent reviews of this subject. The symplectic structure  $B$  is a nondegenerate, closed 2-form, i.e.  $dB = 0$  [21]. Therefore the symplectic structure  $B$  defines a bundle isomorphism  $B : TM \rightarrow T^*M$  by  $X \mapsto A = \iota_X B$  where  $\iota_X$  is an interior product with respect to a vector field  $X \in \Gamma(TM)$ . One can invert this map to obtain the inverse map  $\theta \equiv B^{-1} : T^*M \rightarrow TM$  defined by  $\alpha \mapsto X = \theta(\alpha)$  such that  $X(\beta) = \theta(\alpha, \beta)$  for  $\alpha, \beta \in \Gamma(T^*M)$ .

<sup>1</sup> Here we refer to a background independent theory where any spacetime structure is not *a priori* assumed but defined by the theory.

<sup>2</sup> From now on, we will work in Euclidean space though we still use the term “spacetime”. After illuminating how a space is emergent from  $U(1)$  gauge fields, we will speculatively touch the issue of emergent time.

The bivector  $\theta \in \Gamma(\Lambda^2 TM)$  is called a Poisson structure of  $M$  which defines a bilinear operation on  $C^\infty(M)$ , the so-called Poisson bracket, defined by

$$\{f, g\}_\theta = \theta(df, dg) \quad (5)$$

for  $f, g \in C^\infty(M)$ . Then the real vector space  $C^\infty(M)$ , together with the Poisson bracket  $\{-, -\}_\theta$ , forms an infinite-dimensional Lie algebra, called a Poisson algebra  $\mathcal{P} = (C^\infty(M), \{-, -\}_\theta)$ . First note that the orthonormal tangent vectors  $E_A = E_A^M(x)\partial_M \in \Gamma(TM)$  satisfy the Lie algebra

$$[E_A, E_B] = -f_{AB}^C E_C. \quad (6)$$

In general, the composition  $[X, Y]$ , the Lie bracket of  $X$  and  $Y$ , on  $\Gamma(TM)$ , together with the real vector space structure of  $\Gamma(TM)$ , forms a Lie algebra  $\mathcal{V} = (\Gamma(TM), [-, -])$ . There is a natural Lie algebra homomorphism between the Lie algebra  $\mathcal{V} = (\Gamma(TM), [-, -])$  and the Poisson algebra  $\mathcal{P} = (C^\infty(M), \{-, -\}_\theta)$  defined by [21]

$$C^\infty(M) \rightarrow \Gamma(TM) : f \mapsto X_f \quad (7)$$

such that

$$X_f(g) = \theta(df, dg) = \{f, g\}_\theta \quad (8)$$

for  $f, g \in C^\infty(M)$ . It is easy to prove the Lie algebra homomorphism

$$X_{\{f, g\}_\theta} = [X_f, X_g] \quad (9)$$

using the Jacobi identity of the Poisson algebra  $\mathcal{P}$ .

Let us take  $M = \mathbf{R}^4$  and a constant symplectic structure  $B = \frac{1}{2}B_{MN}dx^M \wedge dx^N$ , for simplicity. A remarkable point is that the electromagnetism on a symplectic manifold  $(\mathbf{R}^4, B)$  is completely described by the Poisson algebra  $\mathcal{P} = (C^\infty(M), \{-, -\}_\theta)$  [22, 23, 12]. For example, the action is given by

$$S = \frac{1}{4g_{YM}^2} \int d^4x \{C_A, C_B\}_\theta^2 \quad (10)$$

where

$$C_A(x) = B_{AB}x^B + A_A(x) \in C^\infty(M), \quad A = 1, \dots, 4 \quad (11)$$

are covariant dynamical coordinates describing fluctuations from the Darboux coordinate  $x^A$ , i.e.  $\{x^A, x^B\}_\theta = \theta^{AB}$ , and

$$\begin{aligned} \{C_A(x), C_B(x)\}_\theta &= -B_{AB} + \partial_A A_B - \partial_B A_A + \{A_A, A_B\}_\theta \\ &\equiv -B_{AB} + F_{AB}(x) \in C^\infty(M). \end{aligned} \quad (12)$$

It is clear [5] that the equations of motion as well as the Bianchi identity can be represented only with the Poisson bracket  $\{-, -\}_\theta$ :

$$\{C^B(x), \{C_A(x), C_B(x)\}_\theta\}_\theta = 0, \quad (13)$$

$$\{C_A(x), \{C_B(x), C_C(x)\}_\theta\}_\theta + \text{cyclic} = 0, \quad (14)$$

where

$$\{C_A, \{C_B, C_C\}_\theta\}_\theta = \partial_A F_{BC} + \{A_A, F_{BC}\}_\theta \equiv D_A F_{BC} \in C^\infty(M). \quad (15)$$

A peculiar thing for the action (10) is that the field strength  $F_{AB}$  in (12) is nonlinear due to the Poisson bracket term although it is the curvature tensor of  $U(1)$  gauge fields. Thus one can consider a nontrivial solution of the following self-duality equation

$$F_{AB} = \pm \frac{1}{2} \varepsilon_{AB}^{CD} F_{CD}. \quad (16)$$

In fact, after the canonical Dirac quantization (40) of the Poisson algebra  $\mathcal{P} = (C^\infty(M), \{-, -\}_\theta)$ , the solution of the self-duality equation (16) is known as noncommutative  $U(1)$  instantons [24, 25, 26]. When applying the Lie algebra homomorphism (9) to (12), we get the identity

$$X_{F_{AB}} = [V_A, V_B] \quad (17)$$

where the vector fields  $V_A \equiv X_{C_A} \in \Gamma(TM)$  are obtained by the map (8) from the set of the covariant coordinates  $C_A(x)$  in (11). As a result, the self-duality equation (16) is mapped to the self-duality equation of the vector fields  $V_A$  [22, 23]:

$$[V_A, V_B] = \pm \frac{1}{2} \varepsilon_{AB}^{CD} [V_C, V_D]. \quad (18)$$

Note that the vector fields  $V_A = V_A^M \partial_M$  are divergence free, i.e.,  $\partial_M V_A^M = 0$  by the definition (8) and so preserves a volume form  $\nu$  because  $\mathcal{L}_{V_A} \nu = (\nabla \cdot V_A) \nu = 0$  where  $\mathcal{L}_{V_A}$  is a Lie derivative with respect to the vector field  $V_A$ . Furthermore it can be shown [12] that  $V_A$  can be related to the vierbeins  $E_A$  by  $V_A = \lambda E_A$  with  $\lambda \in C^\infty(M)$  to be determined.

If the volume form  $\nu$  is given by

$$\nu \equiv \lambda^{-2} \nu_g = \lambda^{-2} E^1 \wedge \dots \wedge E^4 \quad (19)$$

or, in other words,  $\lambda^2 = \nu(V_1, \dots, V_4)$ , one can easily check that the triple of Kähler forms for a hyper-Kähler manifold  $M$  is given by [5]

$$J_+^a = \frac{1}{2} \eta_{AB}^a \iota_A \iota_B \nu, \quad J_-^a = -\frac{1}{2} \bar{\eta}_{AB}^a \iota_A \iota_B \nu, \quad (20)$$

where  $\iota_A$  is the interior product with respect to  $V_A$  and  $\eta_{AB}^a$  and  $\bar{\eta}_{AB}^a$  are self-dual and anti-self-dual 't Hooft symbols [27]. One can prove that gravitational instantons satisfying the self-duality equation

$$R_{MNAB} = \pm \frac{1}{2} \varepsilon_{AB}^{CD} R_{MNCD} \quad (21)$$

are hyper-Kähler manifolds, i.e.,  $dJ_+^a = 0$  or  $dJ_-^a = 0$  and vice versa. It is straightforward to prove [5] that the hyper-Kähler conditions  $dJ_+^a = 0$  or  $dJ_-^a = 0$  are precisely equivalent to (18) which can easily be seen by applying to (20) the formula [21]

$$d(\iota_X \iota_Y \alpha) = \iota_{[X, Y]} \alpha + \iota_Y \mathcal{L}_X \alpha - \iota_X \mathcal{L}_Y \alpha + \iota_X \iota_Y d\alpha \quad (22)$$

for vector fields  $X, Y$  and a  $p$ -form  $\alpha$ .

In retrospect, the self-dual Lie algebra (18) was derived from the self-duality equation (16) of  $U(1)$  gauge fields defined on the symplectic manifold  $(\mathbf{R}^4, B)$ . As a consequence,  $U(1)$  instantons on the symplectic manifold  $(\mathbf{R}^4, B)$  are gravitational instantons [28, 29, 22, 23]! We want to emphasize that the emergence of Riemannian metrics from symplectic  $U(1)$  gauge fields is an inevitable consequence of the Lie algebra homomorphism between the Poisson algebra  $\mathcal{P} = (C^\infty(M), \{-, -\}_\theta)$  and the Lie algebra  $\mathcal{V} = (\Gamma(TM), [-, -])$  if the underlying action of  $U(1)$  gauge fields is given by the form (10). Moreover, the equivalence between  $U(1)$  instantons in the action (10) and gravitational instantons turns out to be a particular case of more general duality between the  $U(1)$  gauge theory on a symplectic manifold  $(M, B)$  and Einstein gravity [12, 30], as will be sketched below.

First of all, we draw general results derived from the Lie algebra isomorphism between the Poisson algebra  $\mathcal{P} = (C^\infty(M), \{-, -\}_\theta)$  and the Lie algebra  $\mathcal{V} = (\Gamma(TM), [-, -])$ . Since  $V_A = \lambda E_A \in \Gamma(TM)$  where  $\lambda^2 = \det V_A^M$ , the Riemannian metric (3) is given by

$$\left(\frac{\partial}{\partial s}\right)^2 = \delta^{AB} E_A \otimes E_B = \lambda^{-2} \delta^{AB} V_A \otimes V_B \quad (23)$$

or

$$ds^2 = \delta_{AB} E^A \otimes E^B = \lambda^2 \delta_{AB} V^A \otimes V^B \quad (24)$$

where  $V^A = \lambda^{-1} E^A \in \Gamma(T^*M)$  is a dual basis of  $V_A \in \Gamma(TM)$ . Note that the smooth functions  $C_A(x) \in C^\infty(M)$  ( $A = 1, \dots, 4$ ) in (11) are linearly independent and thus the vector fields  $V_A \in \Gamma(TM)$  defined by (7) are also linearly independent. Accordingly the vector fields  $V_A$  ( $A = 1, \dots, 4$ ) span a full four-dimensional space. In effect, the metric (24) is completely determined by the set (11) of  $U(1)$  gauge fields and it describes a general Riemannian manifold.

So far we did not impose the equations of motion (13) and the Jacobi identity (14) on the metric (23). Eventually we have to impose them because the set (11) of  $U(1)$  gauge fields obey (13) and (14). In order to do that, let us apply the Lie algebra homomorphism (9) again to (15) to yield

$$X_{D_A F_{BC}} = [V_A, [V_B, V_C]] \in \Gamma(TM). \quad (25)$$

It is then straightforward to get the following correspondence [12]

$$D^B F_{AB} = 0 \quad \Leftrightarrow \quad [V^B, [V_A, V_B]] = 0, \quad (26)$$

$$D_A F_{BC} + \text{cyclic} = 0 \quad \Leftrightarrow \quad [V_A, [V_B, V_C]] + \text{cyclic} = 0. \quad (27)$$

Now a critical question is whether the equations of motion (26) for gauge fields together with the Jacobi (or Bianchi) identity (27) can be written as the Einstein equations for the metric (23). A quick notice is that (26) and (27) will end in some equations related to Riemann curvature tensors because they differentiate the metric (23) twice.

To see what they are, recall that, in terms of covariant derivative, the torsion  $T$  and the curvature  $R$  can be expressed as follows [1]

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \quad (28)$$

$$R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z, \quad (29)$$

where  $X, Y$  and  $Z$  are vector fields on  $M$ . Because  $T$  and  $R$  are multilinear differential operators, we get the following relations [31]

$$T(V_A, V_B) = \lambda^2 T(E_A, E_B), \quad (30)$$

$$R(V_A, V_B)V_C = \lambda^3 R(E_A, E_B)E_C. \quad (31)$$

After imposing the torsion free condition  $T(E_A, E_B) = 0$ , it is straightforward, using (30) and (31), to derive the identity below

$$R(E_A, E_B)E_C + \text{cyclic} = \lambda^{-3}([V_A, [V_B, V_C]] + \text{cyclic}). \quad (32)$$

Therefore we immediately see [12] that the Bianchi identity (27) for  $U(1)$  gauge fields is equivalent to the first Bianchi identity for Riemann curvature tensors, i.e.,

$$D_A F_{BC} + \text{cyclic} = 0 \quad \Leftrightarrow \quad R(E_A, E_B)E_C + \text{cyclic} = 0. \quad (33)$$

The mission for the equations of motion (26) is more involved. An underlying idea is to carefully separate the right-hand side of (26) into a part related to the Ricci tensor  $R_{AB}$  and a remaining part. Basically, we are expecting the following form of the Einstein equations

$$D^B F_{AB} = 0 \quad \Leftrightarrow \quad R_{AB} = 8\pi G \left( T_{AB} - \frac{1}{2} \delta_{AB} T \right). \quad (34)$$

After a straightforward but tedious calculation [12], we get a remarkably simple but cryptic result

$$R_{AB} = -\frac{1}{\lambda^2} \left[ g_D^{(+a)} g_D^{(-b)} \left( \eta_{AC}^a \bar{\eta}_{BC}^b + \eta_{BC}^a \bar{\eta}_{AC}^b \right) - g_C^{(+a)} g_D^{(-b)} \left( \eta_{AC}^a \bar{\eta}_{BD}^b + \eta_{BC}^a \bar{\eta}_{AD}^b \right) \right]. \quad (35)$$

To get the result (35), we have defined the structure equation of vector fields  $V_A \in \Gamma(TM)$

$$[V_A, V_B] = -g_{AB}{}^C V_C \quad (36)$$

and the canonical decomposition

$$g_{ABC} = g_C^{(+a)} \eta_{AB}^a + g_C^{(-\dot{a})} \bar{\eta}_{AB}^{\dot{a}}. \quad (37)$$

A notable point is that the right-hand side of (35) consists of purely interaction terms between self-dual and anti-self-dual parts in (37) which is the feature withheld by matter fields only [32, 33]. Incidentally, the self-duality equation (18) can be understood as  $g_C^{(-\dot{a})} = 0$  (self-dual) or  $g_C^{(+a)} = 0$  (anti-self-dual) in terms of (37) and so  $R_{AB} = 0$  in (35), i.e., (18) describes a Ricci-flat manifold. Of course, this is consistent with the fact that a gravitational instanton is a Ricci-flat, Kähler manifold. Nevertheless a unique property of (35) is to contain a nontrivial trace contribution, i.e., a nonzero Ricci scalar, due to the second part which is not existent in Einstein gravity as was recently shown in [33]. By comparing (35) with (34), a surprising content of the energy-momentum tensor was found in [12], which will be discussed in section 4.

We come to the conclusion that general relativity or gravity can emerge from the electromagnetism supported on a symplectic spacetime  $(M, B)$ , which is an interacting theory defined by the action (10). How is it possible to realize the equivalence principle or general covariance, the most important property in the theory of gravity (general relativity), from the  $U(1)$  gauge theory on a symplectic or Poisson manifold? It turns out [11, 12] that the Poisson structure (5) of spacetime admits a novel form of the equivalence principle even for the electromagnetic force, known as the Darboux theorem or the Moser lemma in symplectic geometry, and consequently the electromagnetism on a symplectic spacetime can be realized as a geometrical property of spacetime. In the end, the symplectization of spacetime geometry would be a novel and authentic way to quantize gravity [5, 20].

### 3. Noncommutative geometry and quantum gravity

We have observed that Einstein gravity can be emergent from electromagnetism as long as spacetime admits a symplectic structure and its underlying theory is completely defined by the Poisson algebra  $\mathcal{P} = (C^\infty(M), \{-, -\}_\theta)$ . For instance, an underlying dynamical system for gravity is described by the action (10) which leads to the equations of motion (13). Note that the Jacobi identity (14) is an important property for  $\mathcal{P}$  to be a Poisson algebra and to form a Lie algebra. One can understand the Lie algebra morphism (7) as the adjoint map defined by

$$\text{ad}_f : g \mapsto \{f, g\}_\theta \quad (38)$$

and thence the action of any element on the algebra is a derivation, i.e.,

$$\text{ad}_f(g \cdot h) = (\text{ad}_f g) \cdot h + g \cdot \text{ad}_f h \quad (39)$$

for  $f, g, h \in C^\infty(M)$ . The Jacobi identity of the Poisson algebra  $\mathcal{P}$  is then equivalent to the identity (9) between the operators of the adjoint representation. This identity implies that the map (38) sending each element to its adjoint action is a Lie algebra homomorphism of the original algebra  $\mathcal{P}$  into the Lie algebra  $\mathcal{V} = (\Gamma(TM), [-, -])$  of its derivations. This is a mathematical

basis to explain how gravity is emergent from the electromagnetism on a symplectic manifold  $(M, B)$ .

Using the isomorphism between the Lie algebras  $\mathcal{P}$  and  $\mathcal{V}$ , we showed that a standard (commutative) dynamical system for gravity can be described in terms of vector fields in  $\mathcal{V}$ . Recall that vector fields on a usual (commutative) space are derivations of the algebra  $C^\infty(M)$  of smooth functions on this space. And vector fields, being a global concept, has its noncommutative (NC) generalization, called a derivation of NC algebra. Now we will show how the derivation of NC algebra can be obtained by canonically (à la Dirac) quantizing the Poisson algebra  $\mathcal{P} = (C^\infty(M), \{-, -\}_\theta)$ .

The Dirac quantization of the Poisson algebra  $\mathcal{P} = (C^\infty(M), \{-, -\}_\theta)$  consists of a complex Hilbert space  $\mathcal{H}$  and of a quantization map  $\mathcal{Q}$  to attach to functions  $f \in C^\infty(M)$  on  $M$  operators  $\hat{f} \in \mathcal{A}_\theta$  acting on  $\mathcal{H}$  [34, 35]. The map  $\mathcal{Q} : C^\infty(M) \rightarrow \mathcal{A}_\theta$  by  $f \mapsto \mathcal{Q}(f) \equiv \hat{f}$  should be  $\mathbb{C}$ -linear and an algebra homomorphism:

$$f \cdot g \mapsto \widehat{f \cdot g} = \hat{f} \cdot \hat{g} \quad (40)$$

and

$$f \star g \equiv \mathcal{Q}^{-1}(\mathcal{Q}(f) \cdot \mathcal{Q}(g)) \quad (41)$$

for  $f, g \in C^\infty(M)$  and  $\hat{f}, \hat{g} \in \mathcal{A}_\theta$ . The Poisson structure (5) controls the failure of commutativity

$$[\hat{f}, \hat{g}] \sim i\{f, g\}_\theta + \mathcal{O}(\theta^2). \quad (42)$$

For example, the coordinate generators of  $\mathcal{A}_\theta$  are noncommuting with the Heisenberg algebra relation

$$[x^A, x^B] = i\theta^{AB} \quad (43)$$

where we omit the hat for the coordinate generators for a notational convenience. From the deformation quantization point of view, the NC algebra of operators in  $\mathcal{A}_\theta$  is equivalent to the deformed algebra of functions defined by the Moyal  $\star$ -product (41) which is, according to the Weyl-Moyal map [34, 35], given by

$$\hat{f} \cdot \hat{g} \cong (f \star g)(x) = \exp\left(\frac{i}{2}\theta^{AB}\partial_A^x\partial_B^y\right)f(x)g(y)|_{x=y}. \quad (44)$$

According to the quantization map (40), every expressions in  $\mathcal{P}$  are mapped to corresponding operators in  $\mathcal{A}_\theta$ . For instance, for the symplectic gauge fields in (11), we have the map

$$C_A(x) \in C^\infty(M) \Rightarrow \hat{C}_A(x) = B_{AB}x^B + \hat{A}_A(x) \in \mathcal{A}_\theta \quad (45)$$

and, for the Poisson bracket in (12),

$$\begin{aligned} \{C_A(x), C_B(x)\}_\theta &\Rightarrow -i[\hat{C}_A(x), \hat{C}_B(x)]_\star = -B_{AB} + \partial_A \hat{A}_B - \partial_B \hat{A}_A - i[\hat{A}_A, \hat{A}_B]_\star \\ &\equiv -B_{AB} + \hat{F}_{AB}(x) \in \mathcal{A}_\theta \end{aligned} \quad (46)$$

where  $[\hat{f}, \hat{g}]_\star = \hat{f} \star \hat{g} - \hat{g} \star \hat{f}$ . The quantized action for NC  $U(1)$  gauge fields is then given by

$$\hat{S} = -\frac{1}{4g_{YM}^2} \int d^4x [\hat{C}_A, \hat{C}_B]_\star^2. \quad (47)$$

Similarly, one can lift the adjoint map (7) or (38) to derivations of the NC algebra  $\mathcal{A}_\theta$ :

$$\text{ad}_f^\star : \hat{g} \mapsto -i[\hat{f}, \hat{g}]_\star \quad (48)$$



that satisfies the Leibniz rule, i.e.,

$$\text{ad}_{\hat{f}}^*(\hat{g} \star \hat{h}) = (\text{ad}_{\hat{f}}^* \hat{g}) \star \hat{h} + \hat{g} \star \text{ad}_{\hat{f}}^* \hat{h} \quad (49)$$

for  $\hat{f}, \hat{g}, \hat{h} \in \mathcal{A}_\theta$ . And the Jacobi identity of the  $\star$ -commutator (48) leads to the conclusion that the polydifferential operator on  $\mathcal{A}_\theta$  [36], whose set is denoted as  $\Gamma_\theta(\widehat{TM})$ ,

$$\text{ad}_{\hat{f}}^* \equiv X_{\hat{f}}^* = X_f + \sum_{n=2}^{\infty} \xi_{X_f}^{A_1 \cdots A_n} \partial_{A_1} \cdots \partial_{A_n} \quad (50)$$

is again a derivation of  $\mathcal{A}_\theta$  satisfying the deformed Lie algebra

$$[X_{\hat{f}}^*, X_{\hat{g}}^*] = X_{[\hat{f}, \hat{g}]_\star}^*. \quad (51)$$

It should be noted that the polydifferential operator (50) recovers the usual vector field in the commutative limit  $\theta \rightarrow 0$ . Hence it is obvious that the left-hand side of (51) is a deformation of the ordinary Lie bracket of vector fields. See [36] for an explicit formula up to second order.

It is easy to “quantize” the Lie algebra homomorphism (25) using the above relation (51)

$$X_{\widehat{D}_A \widehat{F}_{BC}}^* = [V_A^*, [V_B^*, V_C^*]] \in \Gamma_\theta(\widehat{TM}) \quad (52)$$

where  $V_A^* \equiv X_{\widehat{C}_A}^* \in \Gamma_\theta(\widehat{TM})$  are generalized vector fields defined by (50). Accordingly we have a NC generalization of the correspondence (26) given by [12, 5]

$$\widehat{D}^B \widehat{F}_{AB} = 0 \quad \Leftrightarrow \quad [V_B^*, [V_A^*, V_B^*]] = 0, \quad (53)$$

$$\widehat{D}_A \widehat{F}_{BC} + \text{cyclic} = 0 \quad \Leftrightarrow \quad [V_A^*, [V_B^*, V_C^*]] + \text{cyclic} = 0. \quad (54)$$

Since the leading order in (50) recovers the usual vector fields, the Einstein equations (34) will appear as the leading order of NC gauge fields described by (53) and (54) and higher order terms will generate derivative corrections of the Einstein gravity. Although there is no concrete verification so far, it was conjectured in [11] that the resulting emergent gravity from NC gauge fields will be based on the NC geometry defined by

$$\widehat{T}(X^*, Y^*) = \widehat{\nabla}_{X^*} Y^* - \widehat{\nabla}_{Y^*} X^* - [X^*, Y^*], \quad (55)$$

$$\widehat{R}(X^*, Y^*) Z^* = [\widehat{\nabla}_{X^*}, \widehat{\nabla}_{Y^*}] Z^* - \widehat{\nabla}_{[X^*, Y^*]} Z^*, \quad (56)$$

where  $X^*, Y^*, Z^* \in \Gamma_\theta(\widehat{TM})$  and  $\widehat{\nabla}_{X^*}$  is a generalized affine connection on  $\mathcal{A}_\theta$  evaluated at the vector field  $X^*$ . If so, the NC gravity in [37, 38] would be defined by NC gauge fields.

Now we will argue that the NC geometry described by (53) and (54) has to define a quantum gravity at a microscopic scale, e.g., Planck scale  $L_P$  and provides a clue to realize a background independent formulation of quantum gravity [20]. One can meaningfully speak of a NC dynamics (without reference to local concepts such as that of points or time instant) provided that one describes NC dynamics in terms of derivations of the corresponding algebra. This is precisely the geometric basis underpinning the above construction. In this approach, the crux for the background independentness is that the NC spacetime defined by the Heisenberg algebra (43) admits a separable Hilbert space  $\mathcal{H}$  which is an infinite-dimensional Fock space of two-dimensional quantum harmonic oscillators. Therefore any NC fields in  $\mathcal{A}_\theta$ , which are operators acting on  $\mathcal{H}$ , can be represented in the Fock space  $\mathcal{H}$  as  $N \times N$  matrices where

$N = \dim \mathcal{H} \rightarrow \infty$  [23]. In the end, we have a matrix representation, denoted as  $\mathcal{A}_N$ , for the NC  $U(1)$  gauge theory described by (53) and (54) that is given by [39]

$$\hat{D}^B \hat{F}_{AB} = 0 \quad \Leftrightarrow \quad [\mathbf{C}^B, [\mathbf{C}_A, \mathbf{C}_B]] = 0, \quad (57)$$

$$\hat{D}_A \hat{F}_{BC} + \text{cyclic} = 0 \quad \Leftrightarrow \quad [\mathbf{C}_A, [\mathbf{C}_B, \mathbf{C}_C]] + \text{cyclic} = 0, \quad (58)$$

where  $\mathbf{C}_A = B_{AB}x^B + \mathbf{A}_A \in \mathcal{A}_N$  is a matrix representation in  $\mathcal{H}$  of the NC field  $\hat{C}_A(x) \in \mathcal{A}_\theta$ . Note that the matrix equations of motion (57) can be derived from the 0-dimensional IKKT matrix model [40, 41] whose action is given by

$$S_M = -\frac{1}{4g^2} \text{Tr}[\mathbf{C}_A, \mathbf{C}_B]^2. \quad (59)$$

An underlying picture for the emergent gravity [5] will become clear by recasting the arguments so far with the matrix action (59). The action (59) is zero-dimensional and so it does not assume any kind of spacetime structure. There are only four  $N \times N$  Hermitian matrices  $\mathbf{C}_A$  ( $A = 1, \dots, 4$ ) which are subject to a couple of algebraic relations defined by the right-hand sides of (57) and (58). Therefore, in order to *create a Universe (or any existence)*, first we have to specify a vacuum of the theory where all fluctuations are supported. For consistency, the vacuum should also satisfy (57) and (58). Since the action (59) allows infinitely many solutions even with different topologies, it is not unique but there is a natural “primitive” vacuum defined by

$$\langle \mathbf{C}_A \rangle_{\text{vac}} \equiv \hat{A}_A^{(0)} = B_{AB}x^B \quad (60)$$

where  $B_{AB}$  is a constant matrix of rank 4. The vacuum (60) describes a uniform condensate of NC gauge fields and obeys (57) and (58) if  $x^B \in \mathcal{A}_N$  satisfy the Heisenberg algebra (43). We can also introduce fluctuations over the vacuum (60) which are represented by  $\mathbf{A}_A$ . Because there is a Hilbert space  $\mathcal{H}$  as a representation space of the Heisenberg algebra (43), we can regard the matrices  $\mathbf{C}_A \in \mathcal{A}_N$  as operators in  $\mathcal{A}_\theta$  acting on the Hilbert space  $\mathcal{H}$ . According to the Weyl-Moyal map (44), these adjoint operators are in turn mapped to NC fields represented by (45). It is worthy of remark that the pith for the duality between geometry and algebra originates from the coherent condensation (60) of quantum harmonic oscillators in vacuum, which grants a symplectic structure to the vacuum. It is well-known [17] that the NC algebra  $\mathcal{A}_\theta$  generated by (43) admits a nontrivial inner automorphism whose infinitesimal form is called an inner derivation defined by (48). As a result, the dynamics of fluctuations on the vacuum (43) can always be described by the inner derivations of the algebra  $\mathcal{A}_\theta$ , as was verified in (53) and (54). We showed that their commutative limit is nothing but the Einstein gravity or general relativity.

Now we are ready to disclose the secret nature of spacetime we have posed in the introduction. Because the IKKT matrix model (59) does not assume any prior existence of spacetime from the beginning, in other words, (59) is a background independent theory, it is necessary to define a configuration in the algebra  $\mathcal{A}_\theta$ , for instance, like (60), to generate any kind of spacetime structure, even for flat spacetime. So the question is: What is the spacetime emergent from the vacuum (60) which signifies a uniform condensate of NC gauge fields in vacuum? The definition (48) immediately says that the corresponding vector field  $V_A^{(0)} = \delta_A^B \partial_B$  for the vacuum gauge field  $\hat{A}_A^{(0)}$  is precisely that of flat spacetime, i.e.,  $\langle g_{AB} \rangle_{\text{vac}} = \delta_{AB}$ . See (24) for the metric where  $\lambda^2 = 1$  in this case. Remarkably the vacuum (60) responsible for the flat spacetime is not an empty space unlike general relativity. Instead the flat spacetime is emergent from a uniform condensation of gauge fields in vacuum [12, 5]. Its surprising consequences will be discussed in next section.

#### 4. Emergent spacetime and dark energy

To recapitulate, it was shown that the symplectic structure of spacetime  $M$  leads to an isomorphism between symplectic geometry  $(M, B)$  and Riemannian geometry  $(M, g)$  where the deformations of symplectic structure  $B$  in terms of electromagnetic fields  $F = dA$  are transformed into those of Riemannian metric  $g$ . This approach for quantum gravity allows a background independent formulation which provides a novel and authentic way to quantize gravity [20]. As such, the theory should be formulated in a way that the spacetime geometry arises from a vacuum solution to the field equation of the theory. One should not have to specify a preferred background spacetime in order to write down the field equations. The theory (59) described by large  $N$  matrices is precisely the case. Indeed, the flat spacetime arises from the vacuum solution (60). All other fluctuations over the vacuum are represented by NC  $U(1)$  gauge fields with the action (47), whose commutative limit  $\theta \rightarrow 0$  reduces to the action (10) we will mostly focus on. Only these fluctuations will deform the vacuum geometry as  $\partial_A \rightarrow E_A = \partial_A + h_A^M(x)\partial_M$  or  $\delta_{AB} \rightarrow g_{AB} = \delta_{AB} + h_{AB}$ , according to the map (7). One immediate conclusion is, thus, that a uniform condensation of gauge fields such as the vacuum (60) does *not* gravitate [9, 12] because it is simply responsible for the generation of flat spacetime. A natural question is then what is the dynamical scale of the vacuum condensate (60).<sup>3</sup>

Because gravity emerges from NC  $U(1)$  gauge fields described by the action (10) or (47), the parameters,  $g_{YM}$  and  $\theta = (\frac{1}{B})$ , defining the NC gauge theory should be related to the Newton constant  $G$  in emergent gravity. A simple dimensional analysis leads to the relation [12]

$$\frac{G\hbar^2}{c^2} \sim g_{YM}^2 |\theta|. \quad (61)$$

Then one can immediately estimate the vacuum energy  $\rho_{\text{vac}}$  caused by the condensate (60):

$$\rho_{\text{vac}} \sim \frac{1}{g_{YM}^2} |B_{AB}|^2 \sim g_{YM}^2 M_P^4 \sim 10^{-2} M_P^4 \quad (62)$$

where  $M_P = (8\pi G)^{-1/2} \sim 10^{18} \text{GeV}$  is the Planck mass and  $g_{YM}^2 \sim \frac{1}{137}$ . Finally the emergent gravity reveals a remarkable picture that the condensation of Planck energy in vacuum is actually the origin of flat spacetime. That is to say, the huge Planck energy (62) was simply used to make a flat spacetime. Hence we can conclude that the vacuum energy  $\rho_{\text{vac}} \sim M_P^4$  does not gravitate, which is a tangible difference from Einstein gravity. It is of prime importance that the emergent gravity should not contain a coupling of cosmological constant like  $\int d^4x \sqrt{g} \Lambda$ . This important conclusion may be more strengthened by looking at the definition (7) of emergent metric which is insensitive to a constant vacuum energy because it requires the identity

$$X_{F_{AB}-B_{AB}} = X_{F_{AB}} \in \Gamma(TM) \quad (63)$$

for a constant field strength  $B_{AB}$ . Consequently, the emergent gravity clearly dismisses the cosmological constant problem [9, 2].

As a necessary consequence, the emergent gravity respects the shift symmetry (4). For example, under a shift in the  $B$ -field,  $B \rightarrow B' = B + b$ , with  $b$  a constant antisymmetric tensor, the NC field theory (47) defined in the new background  $\theta' = (\frac{1}{B+b})$  is physically equivalent to that in the old one  $\theta = (\frac{1}{B})$  due to the well-known Seiberg-Witten equivalence [42]. Moreover the Hilbert spaces  $\mathcal{H}(\theta')$  and  $\mathcal{H}(\theta)$  for the Heisenberg algebra (43) are isomorphic to each other.

<sup>3</sup> We do not have an understanding yet how time emerges in this context. Therefore it may be rude to talk about the dynamics, energy, etc. But a qualitative nature of the underlying physics can equally be addressed even in Euclidean space. Hence, for some time, we will reluctantly stay in Euclidean space not to get astray due to the notorious issue of emergent time.

Also the vector fields  $V_A'^{(0)}$  and  $V_A^{(0)}$  determined by  $B'$  and  $B$  backgrounds are equally flat as long as they are constant. This means that the shift symmetry (4) belongs to a global automorphism (a Darboux transformation) of the NC algebra  $\mathcal{A}_\theta$ , i.e.  $\mathcal{A}_{\theta'} \cong \mathcal{A}_\theta$ , which can be interpreted as a global Lorentz transformation [5].

If a flat spacetime emerges from the Planck energy condensation (62) in vacuum,<sup>4</sup> we can draw several interesting implications (though necessarily speculative) which seem to resolve all the puzzles posed in the introduction. First of all, it implies that spacetime will behave like an elastic body with the tension of Planck energy [3]. In other words, gravitational fields generated by the deformations of the background (60) will be very weak because the spacetime vacuum is very solid with a stiffness of the Planck mass. Therefore the dynamical origin of flat spacetime explains the metrical elasticity opposing the curving of space as depicted in Figure 1 and the stunning weakness of gravitational force [2]. Furthermore the emergent spacetime implies that the global Lorentz symmetry, being an isometry of flat spacetime, should be a perfect symmetry up to the Planck scale because the flat spacetime was originated from the condensation of the maximum energy in Nature.

The emergence of spacetime by the vacuum condensate (60) probably also has very interesting implications to cosmology [6]. It is worthwhile to notice that the vacuum algebra (43) describes an extremely coherent condensation because it is the Heisenberg algebra of two-dimensional quantum harmonic oscillator. As a result, the spacetime vacuum (60) should describe a zero-entropy state in spite of the involvement of Planck energy like as the electrical resistance of superconductors is zero because the Cooper pair condensate moves as a coherent quantum mechanical entity. It is very mysterious but it should be the case, because the flat spacetime is a completely empty space from the viewpoint of general relativity and so has no entropy. The thermodynamic laws then suggest that a global time evolution of universe will have an arrow since the entropy of universe has to increase anyway since its birth. It was argued in [6] that the coherent condensate (60) of spacetime “monads” may explain *the arrow of time* in the cosmic evolution of our universe.

One may envisage a spontaneous creation of an exponentially expanding universe. But one could not say that the de Sitter universe was created out of field energy in a preexisting space. If we intend to understand the Planck energy condensation in vacuum as a dynamical process, the time scale for the condensate will be roughly of the Planck time. Thus it is natural to consider that the explosive inflation era that lasted roughly  $10^{-33}$  seconds at the beginning of our universe corresponds to the dynamical process for the instantaneous condensation of vacuum energy  $\rho_{\text{vac}} \sim M_P^4$  to enormously spread out a spacetime. A simple consideration of energy conservation for the condensate leads to the expansion rate

$$\frac{1}{2}V_I^2 - \frac{4\pi G\rho_{\text{vac}}}{3}R^2 = 0 \quad \Rightarrow \quad V_I = H_I R \quad (64)$$

where  $H_I = \sqrt{\frac{8\pi G\rho_{\text{vac}}}{3}} \sim M_P$ . This implies that the dynamical process for the vacuum condensate may explain a cosmic inflation to generate an extremely large spacetime  $\sim e^{60}L_H$ . Unfortunately, it is not clear how to microscopically describe this dynamical process by using the matrix action (59).<sup>5</sup> Nevertheless, it is quite obvious that the cosmic inflation should be a

<sup>4</sup> We may emphasize that it is an inevitable consequence if quantum gravity should be formulated in a background independent way and so the spacetime geometry emerges from a vacuum configuration of some fundamental ingredients in the theory. It is then reasonable to assume that the gravitational constant  $G = M_P^{-2}$  would set a natural dynamical scale for the emergence of gravity and spacetime.

<sup>5</sup> Recently there was a very interesting work [43, 44] addressing this issue using the Monte Carlo analysis of the type IIB matrix model in Lorentzian signature. In this work it was found that three out of nine spatial directions start to expand at some critical time after which exactly 3+1 dimensions dynamically become macroscopic.

dynamical condensation in vacuum for the generation of spacetime according to our emergent gravity picture [6].

Note that the vacuum condensate (60) causes the microscopic spacetime to be NC and so introduces a spacetime uncertainty relation. Therefore, a further accumulation of energy over the NC spacetime (43) will be subject to the UV/IR mixing [45]. This spacetime exclusion will prevent a very localized energy from further condensing into the vacuum, which may correspond to the stability of spacetime. This reasoning implies that the condensation of the vacuum energy (62) happened at most only once. By the same reason, the cosmic inflation should take place only once and, thereby, eternal inflation and cyclic universe seem to be inconsistent with our picture [6].

The special relativity unifies space and time into a single entity – spacetime. Hence it will be desirable to put space and time on an equal footing. If a space is emergent, so should time. But the concept of time is more stringent since it is difficult to give up the causality and unitarity. We believe that a naive introduction of NC time, e.g.,  $[t, x] = i\theta$ , will be problematic because it is impossible to keep the locality in time with the NC time and so to protect the causality and unitarity. How can we define the emergent time together with the emergent space? How is it entangled with the space to unfold into a single entity – spacetime and to take the shape of Lorentz covariance? A decisive lesson comes from quantum mechanics.

Quantum mechanics is the formulation of mechanics in NC phase space

$$[x^i, p_j] = i\hbar\delta_j^i. \quad (65)$$

In quantum mechanics, the time evolution of a dynamical system is described by the quantum Hamilton's (Heisenberg) equation

$$i\hbar \frac{d}{dt} \hat{f}(t) = [\hat{f}(t), \hat{H}] \quad (66)$$

where  $\hat{f}(t), \hat{H} \in \mathcal{A}_\hbar$  are operators in the set of observables  $\mathcal{A}_\hbar$ . This equation can be integrated to give a finite time evolution given by

$$\hat{f}(t) = U(t)^\dagger \hat{f}(0) U(t) \quad (67)$$

where  $U(t) = \exp(-i\hat{H}t/\hbar)$  is the time evolution operator. Therefore, as we know well, an intrinsic (particle) time in quantum mechanics is defined as an inner automorphism of NC algebra  $\mathcal{A}_\hbar$ . Note that the time evolution (67) is meaningful only if the underlying algebra  $\mathcal{A}_\hbar$  is NC but it does not require an operator or NC time to describe the history of the particle system.

A notable point is that any Poisson manifold  $(M, \theta)$  always admits a dynamical Hamiltonian system on  $M$  where the Poisson structure  $\theta$  is a bivector in  $\Gamma(\Lambda^2 TM)$  and the dynamics of a system is described by the Hamiltonian vector field  $X_f = \theta(df)$  of an underlying Poisson algebra [21]. After a (Dirac or deformation) quantization of the Poisson algebra, one can describe the dynamics of the system in terms of derivations of an underlying NC algebra. For example, in the classical limit  $\hbar \rightarrow 0$ , the Heisenberg equation (66) reduces to the classical Hamilton's equation

$$\frac{d}{dt} f(t) = X_H(f(t)) = \{f(t), H\}_\hbar \quad (68)$$

where  $X_H$  is a Hamiltonian vector field defined by  $\iota_{X_H}\omega = dH$  with the symplectic structure  $\omega = \sum_i dx^i \wedge dp_i$  of the particle phase space. In order to get an insight about the emergent time, it would be worthwhile to realize that the mathematical structure of emergent gravity is

basically the same as quantum mechanics. The former is based on the NC space (43) while the latter is based on the NC phase space (65).

Therefore we can also apply the same philosophy to the case of NC spacetime [12]. We suggest the concept of time in emergent gravity as the corresponding Hamilton's equation in the NC space (43)

$$\frac{d}{dt}\hat{f}(t) = -i[\hat{f}(t), \hat{H}]_\star \quad (69)$$

where  $\hat{f}(t), \hat{H} \in \mathcal{A}_\theta$ . Its finite version will be given by an inner automorphism, similar to (67), of the NC algebra  $\mathcal{A}_\theta$ . In the commutative limit  $\theta \rightarrow 0$ , (69) reduces to the similar equation as (68)

$$\frac{d}{dt}f(t) = X_H(f(t)) = \{f(t), H\}_\theta \quad (70)$$

where  $X_H$  is a Hamiltonian vector field defined by  $\iota_{X_H}B = dH$ . Conversely, the Heisenberg equation (69) is the quantization of the classical Hamilton's equation (70) following the rule (39). But, if the Hamiltonian  $\hat{H}$  is time-independent, one can infer by analogy with quantum mechanics that the Heisenberg equation (69) describes only an internal time evolution over the space.

Let us consider a dynamical evolution described by the change of a symplectic structure from  $\omega_0 = B$  to  $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$  for all  $0 \leq t \leq 1$  where  $\omega_1 - \omega_0 = dA$ . A remarkable point (due to the Moser lemma [21]) is that there exists a one-parameter family of diffeomorphisms  $\phi : M \times \mathbf{R} \rightarrow M$  such that  $\phi_t^*(\omega_t) = \omega_0$ ,  $0 \leq t \leq 1$ . Then the evolution of the symplectic structure is locally described by the flow  $\phi_t$  starting at  $\phi_0 = \text{identity}$  and generated by time dependent vector fields  $X_t = \frac{d\phi_t}{dt} \circ \phi_t^{-1}$  satisfying the equation

$$\iota_{X_t}\omega_t + A = 0. \quad (71)$$

Actually the covariant coordinates  $X^A(x) = \theta^{AB}C_B(x) = x^A + \theta^{AB}A_B(x)$  in (11) correspond to the Darboux transformation  $\phi_1 : x^A \mapsto X^A$  obeying  $\phi_1^*(B + F) = B$  [11]. Note that the emergence of gravity originates from the global existence of the one-parameter family of diffeomorphisms  $\phi_t \in \text{Diff}(M)$  describing the local deformation of an initial symplectic structure  $\omega_0 = B$  due to the electromagnetic force  $F = dA$ . Here we observe that the fluctuation of background geometry (determined by  $\omega_0 = B$ ) due to the deformation of symplectic structures necessarily accompanies the time evolution of entire geometry.

There are two facts known in symplectic geometry material to the concept of emergent time. The time evolution of a time-dependent system can again be defined by the inner automorphism of an extended phase space whose extended Poisson bivector is given by [21]

$$\tilde{\theta} = \theta + \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial H}. \quad (72)$$

As usual, the generalized Hamiltonian vector field is defined by

$$\tilde{X}_H = -\tilde{\theta}(dH) = \theta^{AB} \frac{\partial H}{\partial x^B} \frac{\partial}{\partial x^A} + \frac{\partial}{\partial t} \quad (73)$$

and the corresponding Hamilton's equation is given by

$$V_0(f) \equiv \tilde{X}_H(f) = \{C_0, f\}_{\tilde{\theta}} = \frac{\partial f}{\partial t} + \{C_0, f\}_\theta \quad (74)$$

where  $C_0 \equiv -H \in C^\infty(M \times \mathbf{R})$ . Basically we regard  $C_0 = -H$  as an another dynamical variable and so we have a set of coordinates denoted by  $\tilde{C}_{\tilde{A}} = (C_0, C_A)$  in  $M \times \mathbf{R}$  and, following

the same philosophy as (8), we have introduced a vector field  $V_0 = \tilde{X}_H \in \Gamma(T(M \times \mathbf{R}))$ . But note that the original vector fields  $V_A \equiv X_{C_A}$  remain intact because of the relation  $\tilde{V}_A = \tilde{\theta}(dC_A) = \theta(dC_A) = V_A$ . Another important point is the theorem of Souriau and Sternberg [46] stating that a nontrivial time evolution in the presence of electromagnetic fields can be described by the Hamilton's equation with a free Hamiltonian  $H = H_0$  but with a new Poisson structure  $\Theta = \left(\frac{1}{B+F}\right)$  deformed by the electromagnetic force  $F = dA$ . In the case of (74), this theorem means [5] that

$$V_0(f) = \frac{\partial f}{\partial t} - \{H_0, f\}_\Theta \quad (75)$$

where  $\Theta^{AB} = \{X^A, X^B\}_\theta$  and  $H_0$  is a Hamiltonian function when  $F = 0$ , viz., for flat spacetime.

The result (75) reveals a consistent picture with general relativity about time [12]. If  $\Theta^{AB}$  is constant (homogeneous), e.g.  $\Theta^{AB} = \theta^{AB}$ , a clock will tick everywhere at the same rate because (75) is exactly the same as the time evolution on flat spacetime. But, if  $\Theta^{AB}(x)$  is not constant (inhomogeneous) and so an underlying geometry is curved, the time evolution will not be uniform and a clock will tick at the different rate at different places. Also it is quite plausible that the local Lorentz symmetry would be recovered on a local chart because  $\Theta^{AB}(x)$  on the local chart will not be significantly changed and thus the time evolution there is locally the same as the flat spacetime.

The above picture can be more illuminated by evaluating the metric on  $M \times \mathbf{R}$  defined by the vector fields  $(V_0, V_A) \in \Gamma(T(M \times \mathbf{R}))$  or their dual one-forms  $(V^0, V^A) \in \Gamma(T^*(M \times \mathbf{R}))$ . The resulting (4+1)-dimensional metric is given by

$$ds^2 = \lambda^2 \left( -dt^2 + V_M^A V_N^A (dx^M - \mathbf{A}^M)(dx^N - \mathbf{A}^N) \right) \quad (76)$$

where  $\mathbf{A}^M = -\left(\theta^{MN} \frac{\partial C_0}{\partial x^N}\right)dt$  and  $\lambda^2 = \nu(V_0, V_1, \dots, V_4)$  with volume form  $\nu = dt \wedge d^4x$ . One can easily see that the metric (76) reduces to the (4+1)-dimensional Minkowski spacetime after turning off all fluctuations.<sup>6</sup> Interestingly, the metric (76) appears as an emergent geometry of matrix quantum mechanics – the BFSS matrix model [47] – whose action is given by

$$S_{MQM} = \frac{1}{g^2} \int dt \text{Tr} \left( -\frac{1}{2} (\mathbf{D}_0 \mathbf{C}_A)^2 + \frac{1}{4} [\mathbf{C}_A, \mathbf{C}_B]^2 \right) \quad (77)$$

where  $\mathbf{D}_0 \mathbf{C}_A = \frac{\partial \mathbf{C}_A}{\partial t} - i[\mathbf{A}_0, \mathbf{C}_A]$ . Using the relationship between large  $N$  matrix model and NC field theory under the Moyal vacuum (60) with  $\langle \mathbf{A}_0 \rangle_{\text{vac}} = 0$ , one can show [23] that the matrix quantum mechanics (77) is equivalent to (4+1)-dimensional NC  $U(1)$  gauge theory. From the NC gauge theory representation of the matrix model (77), it is straightforward to reproduce the emergent metric (76) using the vector fields  $V_A \in \Gamma(TM)$  determined by NC fields. In this way, we may get some deep insight about the formidable issue of emergent time [5].

Now let us return to the result (35) to discuss a content of the energy-momentum tensor defined by its right-hand side. First it will be convenient to decompose the right-hand side (35) into two parts [12]:

$$8\pi GT_{AB}^{(M)} = -\frac{1}{\lambda^2} \left( g_{ACD} g_{BCD} - \frac{1}{4} \delta_{AB} g_{CDE} g_{CDE} \right), \quad (78)$$

$$8\pi GT_{AB}^{(L)} = \frac{1}{2\lambda^2} \left( \rho_A \rho_B - \Psi_A \Psi_B - \frac{1}{2} \delta_{AB} (\rho_C^2 - \Psi_C^2) \right), \quad (79)$$

<sup>6</sup> We do not understand why the time emerges with an opposite sign, i.e., with the Minkowski signature. The signature is just our wishful choice.

where  $\rho_A \equiv g_{BAB}$  and  $\Psi_A \equiv -\frac{1}{2}\varepsilon^{ABCD}g_{BCD}$ . Recall that the result (35) was obtained in Euclidean space. In order to get a corresponding result in (3+1)-dimensional Lorentzian spacetime just like above, we need to start with a three-dimensional NC space. But we cannot complete a full three-dimensional NC space with the Moyal algebra (43) since it is possible only with even dimensions. Instead, it may be necessary to have a Lie algebra vacuum [30], e.g.  $[x^A, x^B] = i\varepsilon^{ABC}x^C$  ( $A, B, C = 1, 2, 3$ ), or a Nambu vacuum, i.e.  $[x^A, x^B, x^C] = i\varepsilon^{ABC}$ . Unfortunately, the calculation for these cases is much more difficult. Even it is quite demanding to define derivations (i.e., vector fields) for the latter case although the former case is rather well-known from the representation theory of Lie algebra. Therefore we will take a simple-minded recipe – the Wick rotation. (We are not happy with this trick.)

The Wick rotation will be defined by  $x^4 = ix^0$ . Under this Wick rotation, we get the results  $\delta_{AB} \rightarrow \eta_{AB}$ ,  $\varepsilon^{1234} = 1 \rightarrow -\varepsilon^{0123} = -1$  and  $\Psi_A \rightarrow i\Psi_A$  in Minkowski spacetime. There are some reasons that the energy-momentum tensor (78) has to be mapped to the one of the usual Maxwell theory in commutative spacetime. Indeed it was argued in [12] that it can be done by reversing the map (7). But, as we already remarked in section 2, the engrossing part is (79) since it is absent in Einstein gravity and would be a rather unique feature of emergent gravity. Since we are eventually interested in a long-wavelength limit, we will take only the scalar mode in (79) which will be a source of the expansion/contraction of spacetime. This will be a good approximation since the remaining term corresponds to a quadruple mode which gives rise to the shear distortion of spacetime and can thus be neglected. For the same reason, we can ignore the Maxwell energy-momentum tensor (78) since it is also a purely quadruple mode. Hence, in the long-wavelength limit where  $\langle \rho_M \rho_N \rangle \simeq \frac{1}{4}g_{MN}\rho_P^2$  and  $\langle \Psi_M \Psi_N \rangle \simeq \frac{1}{4}g_{MN}\Psi_P^2$ , the energy-momentum tensor (79) simply reduces to [12, 5]

$$T_{MN}^{(L)} \simeq -\frac{R}{32\pi G}g_{MN} \quad (80)$$

where  $R = \frac{1}{2\lambda^2}(\rho_M \rho_N + \Psi_M \Psi_N)g_{MN}$  is the Ricci-scalar of the metric (24). Note that  $\lambda^2 = \sqrt{-g}$  and  $\rho_M = \partial_M \lambda^2$ . Since  $\Psi_M$  is in some sense Hodge-dual to  $\rho_M$ , we expect that their fluctuations will be of the same order with a characteristic wavelength  $L_H$ . Therefore, by a simple dimensional argument, we estimate that  $R \sim \frac{1}{L_H^2}$ . In the end, the energy-momentum tensor (80) proves to be<sup>7</sup>

$$T_{MN}^{(L)} \sim -\frac{1}{L_P^2 L_H^2}g_{MN}. \quad (81)$$

We will argue [9, 10] that the weird energy (81) would be originated from “vacuum fluctuations” with the largest possible wavelength  $L_H$  due to the UV/IR mixing triggered by the noncommutativity of spacetime [45]. Recall that the vacuum energy  $\rho_{\text{vac}} \sim M_P^4$  does not couple to gravity since it was used to create a flat spacetime in our picture. Thus vacuum fluctuations over the vacuum (60) will be a leading contribution to the deformation of spacetime curvature. So let us calculate the vacuum fluctuation energy by the action (47) with the largest possible wavelength  $L_H$ :

$$\begin{aligned} \rho &= \rho_{\text{vac}} + \delta\rho \\ &= \frac{1}{4g_{YM}^2} \left( B_{AB} - \hat{F}_{AB}(x) \right)^2 = \frac{1}{4g_{YM}^2} B_{AB}^2 \left( 1 + \theta \hat{F}(x) \right)^2 \end{aligned}$$

<sup>7</sup> In the Lorentzian signature, the sign of the Ricci scalar  $R$  depends on whether fluctuations are spacelike ( $R > 0$ ) or timelike ( $R < 0$ ) [12, 5]. In consequence the spacelike perturbations act as a repulsive force whereas the timelike ones act as an attractive force. When considering the fact that the fluctuations in (79) are random in nature and we are living in (3+1) (macroscopic) dimensions, the ratio of the repulsive and attractive components will end in  $\frac{3}{4} : \frac{1}{4} = 75 : 25$ . Is it outrageous to conceive that this ratio curiously coincides with the dark composition of our universe ?



$$\sim M_P^4 \left(1 + \frac{L_P^2}{L_H^2}\right)^2 \sim M_P^4 + \frac{1}{L_P^2 L_H^2}, \quad (82)$$

where it is natural to assess that  $\hat{F}(x) \sim \frac{1}{L_H^2}$  for a dimensional reason. Note that the vacuum fluctuation energy

$$\delta\rho \sim \frac{1}{2g_{YM}^2} B_{AB} \hat{F}^{AB}(x) \sim \frac{1}{L_P^2 L_H^2} \quad (83)$$

is a total derivative term and so a boundary term on a hypersurface of radius  $L_H$ .<sup>8</sup> If we assume  $L_H$  to be the size of cosmic horizon, the vacuum fluctuation energy (83) is in good agreement with the observed value of current dark energy  $\rho_{DE} \sim (10^{-3}\text{eV})^4$  [12].

Although our conclusion about the dark energy may be too speculative and so a full-fledged formulation is further required, we believe that the emergent gravity from NC gauge fields has plenty of rooms to explain the nature of dark energy and our underlying arguments must be true even in a full-fledged theory.

## 5. A novel unification in noncommutative spacetime

It has been hoped that a physically viable theory of quantum gravity would unify into a single consistent model all fundamental interactions and describe all known observable interactions in the universe, at both subatomic and astronomical scales. We have argued that the gravitation can emerge from NC gauge fields and a background independent quantum gravity can be defined by quantizing spacetime itself. The upshot is that if gravity is emergent, then the spacetime should be emergent too. If so, every structures supported on the spacetime must also be emergent for an internal consistency of the theory. Hence it should be natural that matter fields as well as non-Abelian gauge fields for weak and strong forces have to be emergent together with the spacetime. Thus an urgent question is the following. How to define matter fields as well as non-Abelian gauge fields describing quarks and leptons in the context of emergent geometry?

In order to figure out an underlying picture for emergent matters [12, 5], it would be useful to start with the Feynman's observation about the electrodynamics of charged particles [48]. In 1948, Feynman got a beautiful idea how to understand electrodynamics in terms of symplectic geometry of particle phase space. Briefly speaking, Feynman asks a question what is the most general form of interactions consistent with particle dynamics defined in the quantum phase space (65). Surprisingly he ends up with the electromagnetic force. In other words, the electromagnetic force is only a consistent interaction with a quantum particle satisfying the commutation relation (65). But the Feynman's observation raises a curious question. We know that, beside the electromagnetic force, there exist other interactions, weak and strong forces, in Nature. Thus the problem is how to incorporate the weak and strong forces together into the Feynman's scheme. Because he started with only a few very natural axioms, there seems to be no room to relax his postulates to include the weak and strong forces except introducing extra dimensions. Remarkably it works with extra dimensions!

To be more precise, consider a particle dynamics defined on  $\mathbf{R}^3 \times F$  with an internal space  $F$  whose coordinates are  $\{x^i : i = 1, 2, 3\} \in \mathbf{R}^3$  and  $\{Q^I : I = 1, \dots, n^2 - 1\} \in F$ . The dynamics of the particle carrying an internal charge in  $F$  is defined by a symplectic structure on  $T^*\mathbf{R}^3 \times F$  whose commutation relations are given by [49, 50]

$$[x^i, x^j] = 0, \quad m[x^i, \dot{x}_j] = i\hbar\delta_j^i, \quad (84)$$

$$[Q^I, Q^J] = i\hbar f^{IJK} Q^K, \quad (85)$$

$$[x^i, Q^I] = 0. \quad (86)$$

<sup>8</sup> Unfortunately, the significance of boundary terms such as (83) due to the UV/IR mixing was overlooked in [12, 5]. This boundary contribution caused by the UV/IR mixing could be consistent with the holographic nature of dark energy.

The internal coordinates  $Q^I$  satisfy  $SU(n)$  algebra and so carry their own Poisson structure inherited from the Lie algebra. One more condition, the so-called Wong's equation [51], is implemented by

$$\dot{Q}^I + \frac{1}{2}f^{IJK}(A_i^J Q^K \dot{x}_i + \dot{x}_i A_i^J Q^K) = 0 \quad (87)$$

to ensure that the internal charge  $Q^I$  is parallel-transported along the trajectory of a particle under the influence of non-Abelian gauge fields  $A_i(x, t) = A_i^I(x, t)Q^I$  [52]. Actually, the Wong's equation is the Heisenberg equation (66) for  $Q^I$  with the Hamiltonian  $\hat{H} = \frac{1}{2}(m\dot{x}_i^2 + Q^I Q^I)$ , i.e.,

$$\dot{Q}^I = -\frac{i}{\hbar}[Q^I, \hat{H}]. \quad (88)$$

One can easily show it using the fact  $m\dot{x}_i = p_i - A_i(x, t)$  and  $[p_i, Q^I] = 0$ .

If we repeat the Feynman's question, we can arrive at the conclusion that the most general interaction of a quantum particle on  $\mathbf{R}^3$  carrying an internal charge  $Q^I$  satisfying (87) and the commutation relations (84)-(86) is a non-Abelian interaction of  $SU(n)$  gauge fields [49]. From our perspective, we thus need extra dimensions with the Poisson structure  $F$  satisfying the above commutation relations to realize leptons and quarks interacting with  $SU(2)$  and  $SU(3)$  gauge fields. Wishfully, it will be more desirable to find a mechanism to realize this structure together with the emergence of spacetime geometry.

With this motivation, we consider a  $U(N)$  gauge theory in four dimensions whose action is given by

$$S_{YM} = -\frac{1}{G_s} \int d^4x \text{Tr} \left( \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \Phi_a D^\mu \Phi_a - \frac{1}{4} [\Phi_a, \Phi_b]^2 \right) \quad (89)$$

where  $\Phi_a$  ( $a = 1, \dots, 6$ ) are adjoint scalar fields in  $U(N)$ . For our purpose, we are interested in a large  $N$  limit, in particular,  $N \rightarrow \infty$ . The action (89) is then exactly the bosonic part of 4-dimensional  $\mathcal{N} = 4$  supersymmetric  $U(N)$  Yang-Mills theory, which is the large  $N$  gauge theory of the AdS/CFT correspondence [53, 54, 55]. Suppose that a vacuum of the theory (89) is given by

$$\langle \Phi_a \rangle_{\text{vac}} = B_{ab} y^b, \quad \langle A_\mu \rangle_{\text{vac}} = 0 \quad (90)$$

where  $B_{ab}$  is a constant matrix of rank 6. And assume that the vacuum expectation values  $y^a \in U(N \rightarrow \infty)$  satisfy the algebra

$$[y^a, y^b] = i\theta^{ab} \mathbf{1}_{N \times N} \quad (91)$$

where  $\theta^{ab} = (\frac{1}{B})^{ab}$ . It is then obvious that the vacuum (90) in the  $N \rightarrow \infty$  limit is definitely a solution of the theory (89) and the vacuum algebra (91) is familiar with the Heisenberg algebra of NC space  $\mathbf{R}_\theta^6$ . Consequently the large- $N$  matrices on  $\mathbf{R}^{3,1}$  in the action (89) can be mapped to NC fields in  $C^\infty(\mathbf{R}^{3,1}) \otimes \mathcal{A}_\theta$ .

Let us consider fluctuations  $\hat{A}_M(X) = (\hat{A}_\mu, \hat{A}_a)(x, y)$ ,  $M = 0, 1, \dots, 9$  of the large- $N$  matrices in the action (89) around the vacuum (90)

$$D_\mu(x, y) = \partial_\mu - i\hat{A}_\mu(x, y), \quad \Phi_a(x, y) = B_{ab} y^b + \hat{A}_a(x, y), \quad (92)$$

where the fluctuations are assumed to also depend on the vacuum moduli in (90). Therefore let us introduce 10-dimensional coordinates  $X^M = (x^\mu, y^a)$  and 10-dimensional connections defined by

$$\begin{aligned} D_M(X) &= \partial_M - i\hat{A}_M(X) \\ &\equiv (D_\mu, D_a = -i\Phi_a)(x, y). \end{aligned} \quad (93)$$

As a result, the large- $N$  matrices in the action (89) are now represented by their master fields which are higher-dimensional NC  $U(1)$  gauge fields in (93) whose field strength is given by

$$\hat{F}_{MN} = \partial_M \hat{A}_N - \partial_N \hat{A}_M - i[\hat{A}_M, \hat{A}_N]_\star. \quad (94)$$

In the end, the 4-dimensional  $U(N)$  Yang-Mills theory (89) has been transformed into a 10-dimensional NC  $U(1)$  gauge theory and the action (89) can be recast into the simple form [23]

$$\hat{S}_{10} = -\frac{1}{4g_{YM}^2} \int d^{10}X (\hat{F}_{MN} - B_{MN})^2. \quad (95)$$

To find a gravitational metric dual to the large- $N$  gauge theory (89) or, equivalently, to find an emergent metric determined by the NC gauge theory (95), we can apply the adjoint operation (48) to the 10-dimensional NC gauge fields in (93) after switching the index  $M \rightarrow A = 0, 1, \dots, 9$ :

$$\begin{aligned} \hat{V}_A[\hat{f}](X) &= [D_A, \hat{f}]_\star(x, y) \\ &= V_A^M(x, y) \partial_M f(x, y) + \mathcal{O}(\theta^3) \end{aligned} \quad (96)$$

for  $\hat{f}(x, y) \in C^\infty(\mathbf{R}^{3,1}) \otimes \mathcal{A}_\theta$ . In the commutative limit, the vector fields  $V_A = V_A^M \partial_M \in \Gamma(TM)$  on a 10-dimensional Lorentzian manifold  $M$  is given by

$$V_A = (\partial_\mu + A_\mu^a \partial_a, D_a^b \partial_b) \quad (97)$$

or their dual basis  $V^A = V_M^A dX^M \in \Gamma(T^*M)$  is given by

$$V^A = (dx^\mu, V_b^a(dy^b - A_\mu^b dx^\mu)), \quad (98)$$

where  $V_a^c D_c^b = \delta_a^b$  and

$$A_\mu^a \equiv -\theta^{ab} \frac{\partial \hat{A}_\mu}{\partial y^b}, \quad D_a^b \equiv \delta_a^b - \theta^{bc} \frac{\partial \hat{A}_a}{\partial y^c}. \quad (99)$$

Hence the 10-dimensional geometry dual to the gauge theory (89) or (95) can easily be determined by [23, 12]

$$\begin{aligned} ds^2 &= \lambda^2 \eta_{AB} V^A \otimes V^B \\ &= \lambda^2 (\eta_{\mu\nu} dx^\mu dx^\nu + \delta_{ab} V_c^a V_d^b (dy^c - \mathbf{A}^c)(dy^d - \mathbf{A}^d)) \end{aligned} \quad (100)$$

where  $\mathbf{A}^a = A_\mu^a dx^\mu$  and the conformal factor is defined by

$$\lambda^2 = \nu(V_0, V_1, \dots, V_9) \quad (101)$$

for a 10-dimensional volume form  $\nu = d^4x \wedge d^6y$  or, more generally,  $\nu = d^4x \wedge \nu_6$ .

It has been known from the AdS/CFT duality [53, 54, 55] that the large- $N$  gauge theory (89) is a nonperturbative formulation of type IIB string theory on  $AdS_5 \times \mathbf{S}^5$  background. We have verified above that the 4-dimensional  $U(N)$  gauge theory (89) gives rise to a 10-dimensional gravity with the metric (100) [23]. We see that the existence of nontrivial gauge fields  $A_\mu(x)$  causes the curving of the original flat spacetime  $\mathbf{R}^{3,1}$  and so the four-dimensional spacetime also becomes dynamical together with an entirely emergent 6-dimensional space. Therefore, the large- $N$  gauge theory (89) almost provides a background independent description of spacetime geometry except the original background  $\mathbf{R}^{3,1}$  whose existence was *a priori* assumed at the outset. We confirm again the important picture [5] that, in order to describe a classical geometry

from a background independent theory, it is necessary to have a nontrivial vacuum defined by a “coherent” condensation of gauge fields, e.g., the vacuum defined by (90).

A remarkable aspect of the large- $N$  gauge theory (89) is that it admits a rich variety of topological objects. So our curiosity is what kind of geometry emerges from such a topological object (according to the map (96)) when the topological solution has been defined by the gauge theory (89) or (95) and what kind of object is materialized in four-dimensional spacetime from the stable solution. We will assert that consolidating some generic features of emergent geometry and the Feynman’s picture about the weak and strong forces leads to a remarkable picture for what matter is. In particular, a matter field such as leptons and quarks may simply arise as a stable localized geometry in extra dimensions, which is a topological object in the defining algebra (NC  $\star$ -algebra) of quantum gravity.

Consider a stable class of time-independent solutions in the action (89) satisfying the asymptotic boundary condition (90). For such kind of solutions, we may forget about time and work in the temporal gauge,  $A_0 = 0$ . Since the adjoint scalar fields asymptotically approach the common limit (90) (which does not depend on  $x^i := \mathbf{x}$ ), we can think of  $\mathbf{R}^3$  as having the topology of a three-sphere  $\mathbf{S}^3 = \mathbf{R}^3 \cup \{\infty\}$ , with the point at infinity included. In particular, the matrices  $\Phi_a(\mathbf{x})$  are nondegenerate along  $\mathbf{S}^3$  and so  $\Phi_a$  defines a well-defined map

$$\Phi_a : \mathbf{S}^3 \rightarrow \text{GL}(N, \mathbf{C}) \quad (102)$$

from  $\mathbf{S}^3$  to the group of nondegenerate complex  $N \times N$  matrices. If this map represents a nontrivial class in the third homotopy group  $\pi_3(\text{GL}(N, \mathbf{C}))$ , the solution (102) will be stable under small perturbations, and the corresponding nontrivial element of  $\pi_3(\text{GL}(N, \mathbf{C}))$  represents a topological invariant [56]. In the stable regime where  $N > 3/2$ , the homotopy groups of  $\text{GL}(N, \mathbf{C})$  or  $U(N)$  define a generalized cohomology theory, known as K-theory  $K(X)$  [57]. For example, for  $X = \mathbf{R}^{3,1}$ , this group with compact support is given by [56]

$$K(\mathbf{R}^{3,1}) = \pi_3(\text{GL}(N, \mathbf{C})) = \mathbf{Z}. \quad (103)$$

Note that the map (102) is contractible to the group of maps from  $\mathbf{S}^3$  to  $U(N)$ .

We now come to the connection with K-theory, via the classic Atiyah-Bott-Shapiro (ABS) construction [58] which relates the Grothendieck groups of Clifford modules to the K-theory of spheres. The ABS isomorphism relates complex and real Clifford algebras to K-theory [59]: Such a relation is somehow expected, given that the periodicity of K-theory is similar to the periodicity of Clifford algebras [31]. Note that the group  $K(X)$  also classifies D-branes in type II superstring theory on a manifold  $X$  [60, 61, 62, 63]. In particular, the RR-charge of type IIB D-branes is measured by the K-theory class of their transverse space, so that  $K(\mathbf{S}^p) = \pi_{p-1}(U(N))$  classifies  $(9-p)$ -branes in type IIB string theory on flat  $\mathbf{R}^{9,1}$  spacetime.

The ABS construction uses the gamma matrices of the Lorentz group  $SO(3,1)$  to construct an explicit generator of the K-theory group (103) [59]. Let  $S_{\pm}$  be two irreducible spinor representations of  $Spin(4)$  and define the gamma matrices  $\Gamma^{\mu} : S_{+} \rightarrow S_{-}$  to satisfy the Dirac algebra  $\{\Gamma^{\mu}, \Gamma^{\nu}\} = 2\eta^{\mu\nu}$ . Let us also introduce the Dirac operator  $\mathcal{D} : \mathcal{H} \times S_{+} \rightarrow \mathcal{H} \times S_{-}$  such that

$$\mathcal{D} = \Gamma^{\mu} p_{\mu} + \cdots \quad (104)$$

where  $p_{\mu} = (\omega, \mathbf{p})$  is a four-momentum and the abbreviation denotes possible higher order corrections in higher energies. Here the Dirac operator (104) is regarded as a linear operator acting on a Hilbert space  $\mathcal{H}$  as well as the spinor vector space  $S_{\pm}$ . The Hilbert space  $\mathcal{H}$  would be possibly much smaller than the primitive Fock space for the Heisenberg algebra (91) because the Dirac operator (104) actually acts on collective (coarse grained) modes of the solution (102) [56].

In order to construct stable topological objects that take values in the K-theory (103), it is natural to consider topological solutions made out of  $\Phi_a(x) \in U(N)$  according to the homotopy map (102). As was shown before, the large- $N$  matrices in  $U(N \rightarrow \infty)$  gauge theory can be described by NC  $U(1)$  gauge fields with the action (95) in higher dimensions. In particular, the adjoint scalar fields  $\Phi_a(x) \in U(N)$  are mapped to NC  $U(1)$  gauge fields in extra dimensions and obey the relation

$$-i[\Phi_a, \Phi_b] = -B_{ab} + \hat{F}_{ab}. \quad (105)$$

Therefore, the topological solutions made out of  $\Phi_a(x) \in U(N)$  will be given by NC  $U(1)$  instantons in four or six dimensions. For instance, in four-dimensional subspace, one can consider NC  $U(1)$  instanton solutions given by [24, 25, 26]

$$\hat{F}_{ab} = \pm \frac{1}{2} \varepsilon_{ab}{}^{cd} \hat{F}_{cd} \quad (106)$$

where  $a, \dots, d = 1, \dots, 4$  and, in six dimensions, one can instead consider NC Hermitian  $U(1)$  instantons defined by [64]

$$\hat{F}_{ab} = \pm \frac{1}{4} \varepsilon_{ab}{}^{cdef} \hat{F}_{cd} J_{ef}, \quad (107)$$

$$J^{ab} \hat{F}_{ab} = 0, \quad (108)$$

where  $J_{ab} = \frac{1}{\kappa} B_{ab}$  is a nondegenerate symplectic matrix defined by the vacuum (90). We showed in section 2 that the NC  $U(1)$  instantons in commutative limit are equivalent to gravitational instantons which are hyper-Kähler manifolds and also called Calabi-Yau 2-folds [22, 23]. Similarly it can be shown [64] that the 6-dimensional NC Hermitian  $U(1)$  instantons satisfying (107) and (108) can be recast into Calabi-Yau 3-folds using the vector fields defined by (96). If we define a gravitational instanton as a Ricci-flat, Kähler manifold, the Calabi-Yau 3-fold corresponds to a 6-dimensional gravitational instanton.

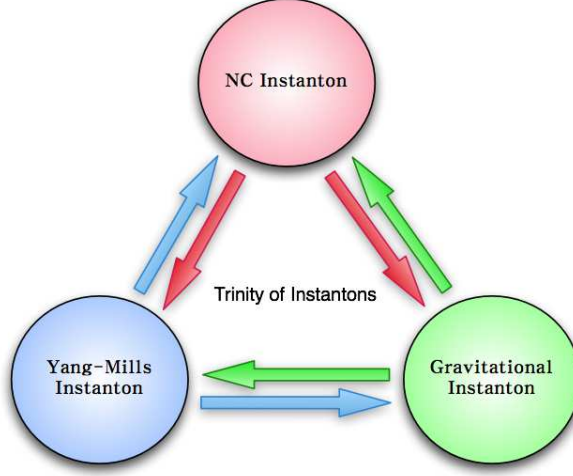
It is well-known that gravity can be formulated as a gauge theory of Lorentz group. In this gauge theory formulation, Calabi-Yau  $n$ -folds correspond to  $SU(n)$  Yang-Mills instantons in  $2n$ -dimensions and the gauge group  $SU(n)$  appears as the holonomy group of a Calabi-Yau  $n$ -fold [27, 65]. Combining the relationship between NC  $U(1)$  instantons,  $SU(n)$  Yang-Mills instantons and Calabi-Yau  $n$ -folds altogether, we get the trinity of instantons [33] depicted in Figure 2.

According to the above construction, the topological solution (102) takes a value in the K-theory group (103) and is realized as a stable geometry, i.e., a Calabi-Yau manifold, in extra dimensions. And the ABS theorem says that the stable topological solution is represented by the Dirac operator (104) on the spacetime  $\mathbf{R}^{3,1}$ . In other words, the Calabi-Yau manifold constructed from a NC  $U(1)$  instanton in extra dimensions will be realized as a four-dimensional fermion  $\chi(t, \mathbf{x})$  whose dispersion relation in low energies is given by the relativistic Dirac equation

$$i\Gamma^\mu \partial_\mu \chi + \dots = 0 \quad (109)$$

as was already suggested by the Dirac operator (104). Although the emergence of 4-dimensional spinors from large- $N$  matrices or NC gauge fields is just a consequence of the fact that the ABS construction uses the Clifford algebra to construct explicit generators of  $\pi_3(U(N))$ , its physical origin is mysterious and difficult to understand.

Recall that the Weyl spinor in (109) is originated from NC  $U(1)$  instantons in extra  $2n$ -dimensions which are also realized as Calabi-Yau  $n$ -folds with  $SU(n)$  holonomy. Therefore the 4-dimensional spinor  $\chi$  should be charged under the  $SU(n) \subset SO(2n)$  gauge group and so can couple to  $SU(n)$  gauge fields in four dimensions. If so, our last question is how non-Abelian gauge fields  $A_\mu^I(x) \in SU(n)$  on  $\mathbf{R}^{3,1}$  can arise from the  $U(1)$  gauge fields on  $\mathbf{R}^{3,1} \times \mathbf{R}_\theta^6$ .



**Figure 2.** Trinity of instantons. (Image from [33])

To recapitulate, the  $U(N \rightarrow \infty)$  gauge theory (89) in the Moyal background (90) has been mapped to the 10-dimensional NC  $U(1)$  gauge theory (95) defined on the space  $\mathbf{R}^{3,1} \times \mathbf{R}_\theta^6$ . Then the K-theory (103) for any sufficiently large  $N$  can be identified with the K-theory  $K(\mathcal{A}_\theta)$  for the NC  $\star$ -algebra  $\mathcal{A}_\theta$ . But, if we consider low-energy excitations around the solution (102) whose K-theory class is given by  $K(\mathcal{A}_\theta)$  and that would be a sufficiently localized state described by a compact (bounded self-adjoint) operator in  $\mathcal{A}_\theta$ , it will not appreciably disturb the ambient gravitational field. This means [12] that we may reduce the problem to a quantum particle dynamics on  $\mathbf{R}^{3,1} \times F$  where  $F$  is an internal space describing collective modes of the solution (102). It is natural to identify the coordinate of  $F$  with an internal charge carried by the Weyl fermion  $\chi$  in (109). We observed above that the (collective) coordinates of  $F$  will take values in the  $SU(n)$  Lie algebra such as the isospins or colors and will be denoted by  $Q^I$  ( $I = 1, \dots, n^2 - 1$ ).

It is useful to remember that the vacuum algebra (91) is nothing but the 3-dimensional quantum harmonic oscillators with the Heisenberg algebra

$$[y^a, y^b] = i\theta^{ab} \quad \Leftrightarrow \quad [a_i, a_j^\dagger] = \delta_{ij} \quad (110)$$

where  $a, b = 1, \dots, 6$  and  $i, j = 1, 2, 3$ . There is a well-known fact that the  $n$ -dimensional harmonic oscillator can realize an  $SU(n)$  symmetry and the generators of the  $SU(n)$  Lie algebra are given by

$$Q^I = \sum_{i,j} a_i^\dagger T_{ij}^I a_j \quad (111)$$

where  $T^I$  are constant  $n \times n$  matrices satisfying the  $SU(n)$  Lie algebra  $[T^I, T^J] = i\hbar f^{IJK} T^K$ . It is easy to check that the Schwinger representation (111) satisfies the commutation relations in (84)-(86). As was reasoned above, the  $SU(n)$  generators in (111) can be regarded as low-energy collective modes (or order parameters) in the vicinity of the solution (102).

Since the chiral fermion  $\chi$  in (109) is charged under the  $SU(n)$  symmetry whose generators are given by (111), it can interact with four-dimensional  $SU(n)$  gauge fields  $A_\mu^I(x)$ . Let  $\rho(\mathcal{H})$  be a representation of the Lie algebra (111). We will take an  $n$ -dimensional representation in  $\mathcal{H} = L^2(\mathbf{C}^n)$  which is much smaller than the original Fock space of (110). Since we are considering a low-energy limit where gravitational back-reactions are ignored, it will be reasonable to take only the lowest modes of NC  $U(1)$  gauge fields  $\hat{A}_\mu(x, y) \in C^\infty(\mathbf{R}^{3,1}) \times \mathcal{A}_\theta$  as

a low-energy approximation. So we will expand the  $U(1)$  gauge fields  $\hat{A}_\mu(x, y)$  in (92) with the  $SU(n)$  basis in (111)

$$\hat{A}_\mu(x, y) = A_\mu(x) + A_\mu^I(x)Q^I + A_\mu^{IJ}(x)Q^IQ^J + \dots \quad (112)$$

where it is assumed that each term in (112) belongs to an irreducible representation of  $\rho(\mathcal{H})$ . Through the expansion (112), we get  $SU(n)$  gauge fields  $A_\mu^I(x)$  as well as ordinary  $U(1)$  gauge fields  $A_\mu(x)$  as low lying excitations [12].

The coarse-grained fermion  $\chi$  in (109) behaves like a stable relativistic particle in the spacetime  $\mathbf{R}^{3,1}$ . Hence, when the particle moves along  $\mathbf{R}^3$ , there will be bosonic excitations arising from changing the position in  $\mathbf{R}^3$  of the internal charge  $F$  according to the relation  $m\dot{x}_i = p_i - A_i(\mathbf{x}, t)$  and the Wong's equation (88). That is, we can think of the Dirac operator (104) as an operator  $\mathcal{H} \times S_+ \rightarrow \mathcal{H} \times S_-$  where  $\mathcal{H} = L^2(\mathbf{C}^n)$  and introduce a minimal coupling with the  $U(1)$  and  $SU(n)$  gauge fields in (112) by the replacement  $p_\mu \rightarrow p_\mu - eA_\mu(x) - A_\mu^I(x)Q^I$ . Then the Dirac equation (109) becomes

$$i\Gamma^\mu \left( \partial_\mu - ieA_\mu - iA_\mu^I(x)Q^I \right) \chi + \dots = 0. \quad (113)$$

Here we see that the chiral fermion  $\chi$  in the homotopy class  $\pi_3(U(N))$  is in the fundamental representation of  $SU(n)$ . As a result, the spinor in (113) can be identified with a quark, an  $SU(3)$  multiplet of chiral Weyl fermions interacting with gluons  $A_\mu^I(x)$  ( $I = 1, \dots, 8$ ) for  $n = 3$  and with a lepton, an  $SU(2)$  doublet of chiral Weyl fermions interacting with isospin gauge fields  $A_\mu^I(x)$  ( $I = 1, \dots, 3$ ) for  $n = 2$  [12, 5].

We want to point out that the emergent matters from stable geometries in extra dimensions are consistent with the Calabi-Yau compactification in string theory. In string theory, a Calabi-Yau manifold serves as an internal geometry of string theory with 6 extra dimensions and their shapes and topology determine a detailed structure of the multiplets for elementary particles and gauge fields through the compactification, which leads to a low-energy phenomenology in four dimensions. A very similar picture seems to be also realized in the context of emergent geometry via the ABS theorem and the trinity of instantons illustrated in Figure 2.

To conclude, we have observed that the theory (89) allows topologically stable solutions as long as the homotopy group (103) is nontrivial. Remarkably, a matter field such as leptons and quarks simply arises from such a stable solution and non-Abelian gauge fields correspond to collective zero-modes of the stable localized solution (102). Although the solution (102) is interpreted as particles and gauge fields ignoring their gravitational effects, we have to recall that it is a stable excitation over the vacuum (90) and so originally a part of spacetime geometry according to the map (96). Consequently, we get a remarkable picture, if any, that matter fields such as leptons and quarks simply arise as a stable localized geometry, which is a topological object in the defining algebra (NC  $\star$ -algebra) of quantum gravity. This approach for quantum gravity thus allows a novel unification where spacetime as well as matter fields is equally emergent from a universal vacuum of quantum gravity [20]. We believe that such an elegant unification of geometry and matters is a unique feature realized only in the background independent formulation of quantum gravity.

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